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Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
fnajman@math.hr

Polynomial Root Motion

Christopher Frayer and James A. Swenson

Abstract. A polynomial is determined by its roots and its leading coefficient. If you set the roots in motion, the critical points will move too. Using only tools from the undergraduate curriculum, we find an inverse square law that determines the velocities of the critical points in terms of the positions and velocities of the roots. As corollaries we get the Polynomial Root Dragging Theorem and the Polynomial Root Squeezing Theorem.

1. INTRODUCTION. Given a polynomial $p(x)$, all of whose roots are real, Rolle's theorem implies that $p(x)$ has exactly one critical point between each pair of adjacent roots. If we move ("drag") some of the roots, the critical points will also change. The Polynomial Root Dragging Theorem (our Corollary 3; see [**1**], [**4**]) explains the change qualitatively: moving one or more roots of the polynomial to the right will cause every critical point to move to the right or stay fixed. Moreover, no critical point moves as far as the root that is moved the farthest.

In this note we consider a more dynamic problem, which we call "polynomial root motion." Rather than forcing all the roots to move in the same direction [**1**], or requiring two roots to be squeezed together (our Corollary 4; see [**2**]), we consider the more general case where each root is allowed to move linearly in a fixed direction. That is, we introduce a time parameter t, let $r_i = a_i + v_i t$ for some constant (possibly zero) velocities v_i, and study

$$p_t(x) = \prod_{i=1}^{n} (x - r_i).$$

The basic question is: at time t_0, which way is a specified critical point moving?

Our Theorem 2, in addition to answering the basic question of polynomial root motion, shows that the roots affect critical points almost like gravity affects masses. Specifically, the influence of a particular root on the velocity of a given critical point varies jointly with the root's velocity and the inverse square of the distance separating the root from the critical point. As corollaries, we obtain independent proofs of the Polynomial Root Dragging Theorem and the Polynomial Root Squeezing Theorem.

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2. POLYNOMIAL ROOT MOTION. Let $p_t(x)$ be a monic polynomial of degree $n \geq 2$, with real, moving roots $r_i = a_i + v_i t$ for $1 \leq i \leq n$, and critical points $c_1(t) \leq c_2(t) \leq \cdots \leq c_{n-1}(t)$. That is,

$$p_t(x) = \prod_{i=1}^{n}(x - r_i) = \sum_{k=0}^{n}(-1)^k \sigma_k x^{n-k}.$$

Here the coefficient σ_k is defined as

$$\sigma_k(r_1, \ldots, r_n) = \sum_{A \subseteq \{1,\ldots,n\},\ |A|=k} \left(\prod_{i \in A} r_i \right),$$

which is the kth elementary symmetric function on $\{r_i \mid 1 \leq i \leq n\}$ (see [**3**, p. 607]). Conventionally, we define $\sigma_0 \equiv 1$ and $\sigma_{-1} \equiv 0$.

For future use, we discuss $\frac{\partial \sigma_k}{\partial r_j}$. Observe that σ_k contains $\binom{n}{k}$ terms, each of which is a product of k distinct factors r_i. Of these $\binom{n}{k}$ terms, $\binom{n-1}{k-1}$ contain r_j. The partial derivative of each of these terms with respect to r_j is obtained simply by deleting the r_j. The other terms of σ_k are constant with respect to r_j; it follows that $\frac{\partial \sigma_k}{\partial r_j} = \sigma_{k-1}(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n)$. For convenience, we introduce the notation $\sigma_k^{\hat{j}} = \sigma_k(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n)$, so $\frac{\partial \sigma_k}{\partial r_j} = \sigma_{k-1}^{\hat{j}}$.

We're now ready to begin work on the basic question. Notice first that the equation $p_t'(x) = 0$ implicitly defines a critical point as a function $x = c_i(t)$. For (x, t) in a neighborhood where $p_t''(x) \neq 0$, $c_i(t)$ is continuously differentiable by the implicit function theorem (see [**5**, Theorem 9.28]).

Let's understand in detail what happens when $p_{t_0}'(x) = p_{t_0}''(x) = 0$. In this case, since p_{t_0} has n real roots, x is a root of p_{t_0} with multiplicity at least three. We'll call this a "triple root collision"—it means that at least two of the $n - 1$ critical points of p_{t_0} "collide" at x. At triple root collisions, $c_i(t)$ is still continuous as it is squeezed between roots that are colliding, and continuity follows from the Squeeze Theorem.

In the lemma that follows, we study an arbitrary critical point of $p_t(x)$. Note that σ_k, $\sigma_k^{\hat{i}}$, and c^k depend implicitly on t.

Lemma 1. *If $c(t)$ is a differentiable function on (a, b) such that $c(t)$ is a critical point of $p_t(x)$ for all $t \in (a, b)$, then*

$$\frac{dc}{dt} = \frac{\sum_{k=0}^{n-2} \left[\left((-1)^k (n-k-1) c^{n-k-2} \right) \sum_{i=1}^{n} v_i \sigma_k^{\hat{i}} \right]}{\sum_{k=0}^{n-2} (-1)^k (n-k)(n-k-1) c^{n-k-2} \sigma_k}$$

wherever this expression is defined.

Proof. We have $p_t(x) = \sum_{k=0}^{n}(-1)^k \sigma_k x^{n-k}$, so c is a critical point of $p_t(x)$ when

$$0 = p_t'(c) = \sum_{k=0}^{n-1}(-1)^k \sigma_k (n-k) c^{n-k-1}.$$

(We have taken advantage of the usual convention that $c^0 \equiv 1$, even when $c = 0$.) Differentiating with respect to t yields

$$0 = \sum_{k=0}^{n-1} \left[(-1)^k \sigma_k (n-k)(n-k-1) c^{n-k-2} \frac{dc}{dt} + (-1)^k \frac{d\sigma_k}{dt}(n-k) c^{n-k-1} \right]. \quad (1)$$

By the chain rule

$$\frac{d\sigma_k}{dt} = \sum_{i=1}^{n} \frac{\partial \sigma_k}{\partial r_i} \frac{dr_i}{dt} = \sum_{i=1}^{n} \sigma_{k-1}^{\hat{i}} v_i,$$

and (1) becomes

$$\frac{dc}{dt} = \frac{\sum_{k=0}^{n-1} (-1)^{k+1} \sum_{i=1}^{n} \sigma_{k-1}^{\hat{i}} v_i (n-k) c^{n-k-1}}{\sum_{k=0}^{n-1} (-1)^k \sigma_k (n-k)(n-k-1) c^{n-k-2}},$$

where we have hypothesized that the denominator is nonzero. In the numerator, the $k = 0$ term vanishes, because $\sigma_{-1}^{\hat{i}} = \frac{\partial}{\partial r_i}(1) = 0$, and in the denominator, the $k = n-1$ term vanishes, because $n - k - 1 = 0$. Reindexing in the numerator, we obtain

$$\frac{dc}{dt} = \frac{\sum_{k=0}^{n-2} \left[\left((-1)^k (n-k-1) c^{n-k-2} \right) \sum_{i=1}^{n} v_i \sigma_k^{\hat{i}} \right]}{\sum_{k=0}^{n-2} (-1)^k (n-k)(n-k-1) c^{n-k-2} \sigma_k},$$

as desired. ■

We pause to analyze a few special cases. First, we analyze the $n = 2$ case, to verify that Lemma 1 is plausible. Here, $\sigma_{n-2} = \sigma_{n-2}^{\hat{j}} = 1$ for $j \in \{1, 2\}$, so we have $c'(t) = (v_1 + v_2)/2$, as we should expect.

Let's also consider the case where all v_i are equal—say $v_i = v$ for all i. In this case, the entire graph is being translated to the right with velocity v. Note that each term of σ_k contains k different factors. Each term is omitted from $\sigma_k^{\hat{i}}$ for exactly k distinct values of i, and thus appears $(n-k)$ times in $\sum_{i=1}^{n} v_i \sigma_k^{\hat{i}}$. It follows that

$$\begin{aligned} \frac{dc}{dt} &= \frac{\sum_{k=0}^{n-2} \left[\left((-1)^k (n-k-1) c^{n-k-2} \right) \sum_{i=1}^{n} v_i \sigma_k^{\hat{i}} \right]}{\sum_{k=0}^{n-2} (-1)^k (n-k)(n-k-1) c^{n-k-2} \sigma_k} \\ &= \frac{\sum_{k=0}^{n-2} \left[\left((-1)^k (n-k-1) c^{n-k-2} \right) v (n-k) \sigma_k \right]}{\sum_{k=0}^{n-2} (-1)^k (n-k)(n-k-1) c^{n-k-2} \sigma_k} = v, \end{aligned} \tag{2}$$

as desired.

We now use Lemma 1 to show that the roots affect critical points almost like gravity affects masses: the influence of a particular root on the velocity of a given critical point varies jointly with the root's velocity and the inverse square of the distance separating the root from the critical point.

Theorem 2 (Polynomial Root Motion Theorem). *Suppose that $c(t)$ is a critical point of $p_t(x)$ for all t, with c differentiable, $c(t_0) = 0$, and $p''_{t_0}(0) \neq 0$. If $p_{t_0}(x)$ has a double root at $x = 0$ (say, $r_k = r_k(t_0) = 0$ if and only if $k \in \{i, j\}$), then $c'(t_0) = (v_i + v_j)/2$. Otherwise,*

$$c'(t_0) = \frac{-p_{t_0}(0)}{p''_{t_0}(0)} \sum_{i=1}^{n} \frac{v_i}{r_i^2}.$$

Proof. The σ_k, at $t = t_0$, are the coefficients of p_{t_0}, up to sign. Hence, by Taylor's theorem, $\sigma_n = (-1)^n p_{t_0}(0)$, $\sigma_{n-1} = (-1)^{n+1} p'_{t_0}(0) = 0$, and $2\sigma_{n-2} = (-1)^n p''_{t_0}(0)$.

Since $\sigma_{n-2} \neq 0$, setting $c = 0$ in Lemma 1 gives

$$c'(t_0) = \sum_{i=1}^{n} \frac{\sigma_{n-2}^{\hat{\imath}}}{2\sigma_{n-2}} v_i .$$

First, suppose that $x = 0$ is a root of p_{t_0}. Since p_{t_0} has n real roots, and 0 is a critical point, 0 must be a multiple root. Since $p''_{t_0}(0) \neq 0$, 0 is a root of multiplicity two, so assume that $r_k = 0$ if and only if $k \in \{i, j\}$. Then $\sigma_{n-2} = \prod_{k \notin \{i,j\}} r_k$, and so

$$\sigma_{n-2}^{\hat{k}} = \begin{cases} \sigma_{n-2} & k \in \{i, j\}, \\ 0 & k \notin \{i, j\}, \end{cases}$$

which yields

$$c'(t_0) = \frac{\sigma_{n-2}}{2\sigma_{n-2}} v_i + \frac{\sigma_{n-2}}{2\sigma_{n-2}} v_j = \frac{v_i + v_j}{2}.$$

Now consider the case where $x = 0$ is not a root of p_{t_0}. Observe that, for each j, $\sigma_j^{\hat{\imath}} = \sigma_j - r_i \sigma_{j-1}^{\hat{\imath}}$. Inductively, we obtain

$$\sigma_{n-2}^{\hat{\imath}} = \sum_{k=0}^{n-2} (-1)^{n-k} \sigma_k r_i^{n-2-k}.$$

It follows that

$$0 = (-1)^n p_{t_0}(r_i) = \sum_{k=0}^{n} (-1)^{n-k} \sigma_k r_i^{n-k} = \sigma_n - r_i \sigma_{n-1} + r_i^2 \sigma_{n-2}^{\hat{\imath}}.$$

Solving for $\sigma_{n-2}^{\hat{\imath}}$, we can rewrite

$$c'(t_0) = \sum_{i=1}^{n} \frac{\sigma_{n-2}^{\hat{\imath}}}{2\sigma_{n-2}} v_i = \sum_{i=1}^{n} \frac{r_i \sigma_{n-1} - \sigma_n}{r_i^2 \cdot 2\sigma_{n-2}} v_i .$$

Using Taylor's formula, as above, we have

$$c'(t_0) = \sum_{i=1}^{n} \frac{r_i \sigma_{n-1} - \sigma_n}{r_i^2 \cdot 2\sigma_{n-2}} v_i = \sum_{i=1}^{n} \frac{-r_i p'_{t_0}(0) - p_{t_0}(0)}{r_i^2 p''_{t_0}(0)} v_i = \frac{-p_{t_0}(0)}{p''_{t_0}(0)} \sum_{i=1}^{n} \frac{v_i}{r_i^2}$$

as desired. ∎

While Theorem 2 has a noticeably simpler statement than Lemma 1, it provides a complete answer to the basic question of polynomial root motion! After all, $p_t(x)$ can always be translated horizontally to place the critical point of interest at $x = 0$ when $t = t_0$, and then Theorem 2 lets us determine which way that critical point is moving at $t = t_0$.

We can read Theorem 2 as representing the velocity of a critical point at a given time as a certain weighted average of the v_i.[1] This suggests that we can use Theorem 2 to give an independent proof of the Polynomial Root Dragging Theorem.

[1] The sum of the weights equals one, as it should; we see this by substituting 1 for v in (2).

Corollary 3 (Polynomial Root Dragging Theorem, [1]). *Let $p(x)$ be a degree-n polynomial with n real roots. When we drag a subset of the roots to the right, its critical points either stay fixed or move to the right.*

Proof. It's enough to prove the theorem for a single critical point, in the case when exactly one root (say, r_1) is moved d units to the right. Define $p_t(x)$ by taking $p_0(x) = p(x)$, $v_1 = d > 0$, and $v_i = 0$ for all $i \neq 1$. Choose a critical point, and let its position at time t be given by $c_i(t)$. There are only finitely many times when r_1 collides with another root. Since c_i is continuous, it will suffice to show that c_i is nondecreasing on each open interval between these times. Consider such an interval. Since r_1 is a single root throughout this interval, $c_i \neq r_1$ in this interval. If c_i coincides with some other root, then there are distinct $j, k \neq 1$ such that $c_i = r_j = r_k$ at some time during the interval. But r_j and r_k are not moving, so $c_i = r_j = r_k$ throughout the interval, and therefore c_i is constant on the interval.

Now suppose that c_i does not coincide with any root at any time in the interval, and let t_0 be a fixed time in the interval. Translating in x as necessary, we may assume that $c_i(t_0) = 0$. Theorem 2 gives

$$c_i'(t_0) = \frac{-p_{t_0}(0)}{p''_{t_0}(0) r_1^2} d.$$

Since 0 is a critical point and $p_{t_0}(0) \neq 0$ is a local extremum, $p_{t_0}(0)/p''_{t_0}(0) < 0$ by the second derivative test, and again $c_i'(t_0) > 0$, as desired. Hence $c_i(0) \leq c_i(1)$, and c_i will move to the right or stay fixed. ∎

Corollary 4 (Polynomial Root Squeezing Theorem, [2]). *Let $p(x)$ be a polynomial of degree n with (possibly repeated) real roots $r_1, r_2, \ldots, r_n$, and select $r_i < r_j$. If r_i and r_j move equal distances toward each other, without passing other roots, then each critical point will stay fixed or move toward $(r_i + r_j)/2$.*

Proof. Fix $d > 0$, the distance which r_i and r_j will move. Define $p_t(x)$ by taking $p_0(x) = p(x)$, $v_i = d$, $v_j = -d$, and $v_k = 0$ for all $k \notin \{i, j\}$. Fix $t_0 \in (0, 1)$, choose a critical point, and let its position at time t be given by $c_k(t)$.

We have assumed that r_i and r_j do not pass other roots. Thus, if $c_k(t_0)$ is at a multiple root, it must be a stationary multiple root; hence c_k is constant. Otherwise, we translate in x to move $c_k(t_0)$ to $x = 0$. Applying Theorem 2, we have

$$c_k'(t_0) = \frac{-d \cdot p_{t_0}(0)}{p''_{t_0}(0)} \left(\frac{1}{r_i^2} - \frac{1}{r_j^2} \right).$$

As we saw in the proof of Corollary 3, $-d \cdot p_{t_0}(0)/p''_{t_0}(0) > 0$. Hence $c_k'(t_0) > 0$ if and only if $|r_i| < |r_j|$. That is, at time t_0, c_k is moving in the same direction as the nearer of $\{r_i, r_j\}$, and is therefore moving toward the more distant of $\{r_i, r_j\}$. This implies the desired conclusion, no matter where $c_k(t_0)$ is located relative to $\{r_i, r_j\}$. ∎

3. CONCLUSION.

We find it particularly satisfying that these results on polynomial root motion can be proved using standard topics from the undergraduate curriculum, such as implicit differentiation, the second derivative test, and Taylor's theorem. It would be nice to prove an analogue of Corollaries 3 and 4 in the case where different subsets of the roots are moved in opposing directions. If $c_i(t)$ is a critical point of $p_t(x)$ for all t, then in principle we can integrate $c_i'(t)$ to find the net change in the critical point. Unfortunately, for this purpose, Lemma 1 does not provide a usable expression for $c_i'(t)$. Nor can we apply the simpler Theorem 2, which is valid only for a single value $t = t_0$.

An interesting exercise is to completely characterize the $n = 3$ case, where

$$y = p_t(x) = A \prod_{i=1}^{3} (x - r_i(t)).$$

It is not hard to explain what happens when roots collide, or to show that a critical point can change directions at most once. When two roots collide, Theorem 2 implies that the critical point between the two roots will move away from the collision in the direction of the fastest moving root. We can describe the triple root collision qualitatively, despite the fact that Theorem 2 does not apply in this case. Indeed, r_1, r_2, r_3, and c are all odd functions of t. The fact that a critical point changes direction at most once, which follows as $\frac{dc}{dt}$ is monotonic when $n = 3$, was a complete surprise to us: we thought that it would be possible to find velocities and initial positions of the roots that would send fast-moving roots shooting past the critical point at different times, from opposite directions, producing at least two changes in direction. What can one say in the degree-n case?

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Department of Mathematics, University of Wisconsin-Platteville, Platteville, WI 53818
frayerc@uwplatt.edu
swensonj@uwplatt.edu

From Enmity to Amity

Aviezri S. Fraenkel

Abstract. Sloane's influential On-Line Encyclopedia of Integer Sequences is an indispensable research tool in the service of the mathematical community. The sequence A001611 listing the "Fibonacci numbers + 1" contains a very large number of references and links. The sequence A000071 for the "Fibonacci numbers −1" contains an even larger number. Strangely, resentment seems to prevail between the two sequences; they do not acknowledge each other's existence, though both stem from the Fibonacci numbers. Using an elegant result of Kimberling, we prove a theorem that links the two sequences amicably. We relate the theorem to a result about iterations of the floor function, which introduces a new game.

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